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ABSTRACT

This study examined the links students are able to make between formal expressions and situations to which they might refer. Students' understanding of a basic set of transformational rules in elementary algebra sign changes in addition and subtraction expressions with parentheses was assessed, exploring: (1) informal intuitive understanding of situations in which instances of the sign-change principles were implicit; (2) knowledge of the formal rules applied to numerical expressions; and (3) ability to link the two. Students from grades 5, 7, and 9 from parochial schools in the Pittsburgh, Pennsylvania area participated; 28 students from each grade participated in the main study. The situations and the interviews are described, followed by the results, which are categorized under informal understanding of principles, sign change rules applied to formal expressions, and linking of expressions and situations. It is concluded that increasing students' understanding of referential meaning of formal symbol systems may facilitate the learning of formal rules and the application of mathematics to problem solving and learning more advanced mathematics. (MNS)

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Referents and Understanding of Algebraic Transformations

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Referents and Understanding of Algebraic Transformations

Transformations of expressions constitute an important part of algebra, and most of the school algebra curriculum is devoted to learning rules for various acceptable transformations. Important work has been done on the difficulties of learning these rules (Adi, 1978; Carry, Lewis, & Bernard, 1980; Davis, Jockusch, & McKnight, 1978; Matz, 1982; Sleeman, 1984; Wagner, Rachlin, & Jensen, 1984) and on ways to improve such learning (e.g., Sweller & Cooper, 1985). We have limited knowledge, however, of how students *understand* the transformations and the expressions they operate on--how they attach meaning to the formalisms.

Our work is guided by the hypothesis that understanding of many mathematical concepts develops by linking mathematical formalism with the real-world situations that children understand intuitively , and then abstracting away from those situations. Situations can be mathematized by expressing their quantitative relationships in an appropriate mathematical formalism (as when writing an equation to solve a problem). The situation can then be thought of as providing referential meaning for the equation. It then becomes possible to use situations to explain and justify the transformations rules of algebra.

But this referential meaning points to an important paradox in learning and using mathematics. On the one hand, formal expressions take their meaning, at least in part, from situations to which they might refer. On the other hand, algebra derives its power by ignoring the specifics of those situations. When algebra is used to solve problems, the problem solver must first mentally represent the situation--the quantities, relationships, and transformations involved--and then write equations that encode those representations in algebraic notation. But facility in using the transformations to solve equations is gained if one ignores the specific situations that generated the equations. As transformations become

more involved, it becomes a practical necessity to ignore the actual situation.

How do students make links between the formalisms of algebra and situations with which they are familiar? Much of the elementary algebra curriculum focuses on the learning of manipulative rules within the formal symbolic system. But students often attempt to learn these rules without linking them to their informal, intuitive understanding of mathematics, reflecting the broader problem of school mathematics learning failing to build upon more informal quantitative knowledge (Ginsburg, 1977; Resnick, in press). Similarly, recent theories of algebra learning (Matz, 1982; Sleeman, 1984) account for errors as deformations of symbol manipulation rules; they involve no representation of the quantities or relationships among quantities.

The current study examined the links students are able to make between formal expressions and situations to which they might refer. We focused on students' understanding of a basic set of transformational rules in elementary algebra: sign changes in addition and subtraction expressions with parentheses [e.g., $a-(b+c)=a-b-c$ and $a-(b-c)=a-b+c$, exploring (a)the children's informal intuitive understanding of situations in which instances of the sign-change principles were implicit, (b)their knowledge of the formal rules applied to numerical expressions, and (c)their ability to link the two.

To understand the sign change transformations intuitively--in terms of actual quantities or specific situations--requires conceiving of the expressions inside parentheses as composed quantities. In the case of 17-(11+4), for example, the total quantity subtracted from 17 is 11+4, or 15. The quantity represented by the entire expression is preserved if, instead of subtracting the sum of 11 and 4, the 11 and the 4 are subtracted separately from the 17, as represented by 17-11-4. In the case of 18-(12-5), the quantity subtracted from 18 is 12-5, or 7. To preserve to quantity represented by this expression, if the 12 and 5 are to be handled separately, one must first subtract the 12, then add back to the original quantity, as represented by 18-12+5.

Method

Subjects

Students from grades 5, 7, and 9 from parochial schools in the Pittsburgh area participated in the study. The main sample consisted of 28 students from each of the three grades. The ninth-graders were enrolled in Algebra I classes. The high school they attended also had slower paced Elementary Algebra classes for students having difficulty and an accelerated Algebra I/II class. The range of students in our ninth-grade sample was thus somewhat restricted. We did interview a small number of students from the slower paced and accelerated classes to supplement our main sample, but these students were not included in our quantitative analyses.

Expressions and Story Situations

Two sets of expressions were used in the interviews. the first set consisted of *a-(b+c)*, its correct transformation, *a-b-c*, and its frequently made incorrect transformation, *a-b+c*. We call this set *parentheses-plus* because of the plus sign inside the parentheses. The second set, *parentheses-minus*, consisted of the expressions *a-(b-c)*, *a-b+c*, and *a-b-c*. To explain why the first two expressions of each set are equivalent, but the first and third are not, requires some understanding of the sign change rules.

Story situations that can be described by the expressions were generated for the parentheses-plus and parentheses-minus sets. Two settings were used for the stories to provide informal domains with which students would likely be familiar: buying items in a store (toys or records) and manipulating sets of discrete objects (cupcakes or cookies). The four sets of story situations are presented In Table 1, with the stories in each set displayed in the order presented in the interview. The situation best described by a key expression [*a-(b+c)* or *a-(b-c)*] was always presented first, with the order of the remaining stories varying across sets. The expressions describing the stories are also presented in Table 1 to clarify their structure; the

formal expressions did not appear in the stories shown to the children during the interviews.

Insert Table 1 about here

Classification schemes for addition and subtraction story problems have been offered by other researchers (e.g., Carpenter & Moser, 1982; Riley, Greeno, & Heller, 1983; Vergnaud, 1982) to describe the conceptual structure of the problems--the relationships among the quantities in the stories. Three commonly used types of problems are *change*, *combine*, and *compare*. Change problems involve actively changing the quantity of a starting set, for example, giving marbles to a person or eating some of a set of apples Combine problems involve static relationships among a set and its subsets. No action is implied. For example, John has 3 red marbles and 4 blue marbles. How many marbles does he have? Compare problems involve static relationships between two separate sets. For example, John has 3 marbles. Betty has 5 marbles. How many more marbles does Betty have?

Because the story situations used in this study involve three quantities, each problem is generally a mixture of types. We restricted the problems to include change and combine relationships because these types resulted in the most natural situations. We did not, however, attempt to create pure change or combine problems; rather, we sought situations that provided the most natural fit with the expressions of interest.

The Interview

Each student was interviewed individually by a single interviewer for approximately 40 minutes. The tasks and questions to be posed by the interviewer were specified in advance, although the interviewers were to be flexible in their use of probing questions to draw out students' understandings and misunderstandings. The interview proceeded in three phases: In Phase 1 the student judged the equivalence of story situations; in Phase 2 the student judged the

equivalence of formal expressions; in Phase 3 the student matched expressions to story situations.

Phase 1: Equivalence of situations. Phase 1 assessed the child's informal, intuitive understanding of situations in which the principles underlying the sign change rules were implicit. The child was presented with one of the story sets shown in Table 1. (Note that the child was *not* presented with the mathematical expressions describing the situations.) The interviewer read the three stories aloud and then asked the child, "Which of the stories are about the same?" After the child selected two stories as being equivalent, the interviewer asked, "Why would those be the same?", following up with probes if needed to clarify the child's justification. The interviewer then asked the child to explain why the remaining, nonequivalent stories were different. This entire procedure was repeated for each of the four situation sets: a parentheses-plus set in the money and discrete object settings, and a parentheses-minus set in each of the two settings. The order in which the story sets were presented was balanced across students to avoid order effects.

Phase 2: Equivalence of formal expressions. Phase 2 assessed the child's knowledge of sign change rules applied to formal expressions. The student was shown pairs of expressions and asked to say whether the expressions in each pair were equal or not equal. The expressions were presented in two sets: parentheses-plus and parentheses minus expressions. In each set, the ***B*** **expression is the** ***A*** **expression without the parentheses [incorrect removal of parentheses]. The** ***C*** **and** ***D*** **expressions include the correct transformation and an incorrect transformation [change the first *minus* to a *plus*].** Half of the students received the parentheses-plus set first; half received the parentheses-minus set first. The parentheses-plus expressions were arranged on a page as follows:

A. 16-(8+3)

B. 16-8+3
C. 16+8-3
D. 16-8-3

The parentheses-minus expressions were

A.	11-(5-2)	B.	11-5-2
		C.	11-5+2
		D.	11+5-2

The interviewer first used cards to cover all the expressions except *A* and *B* and said, "Without calculating or figuring out the answers, tell me if you think these expressions are equal or not equal." After the student said whether the expressions were equal or not equal, the interviewer asked, "Why would these be the same?" or "Why would these be different?" After the student responded, the interviewer repeated the process with expressions *A* and *C*, then with expressions *A* and *D*, keeping only the expressions being compared in view. After the student had made and justified the three judgments, he or she calculated the values of each expression. Then the interviewer again asked the student to explain why the pairs were or were not equivalent.

Phase 3: Linking situations and expressions. Phase 3 of the interview assessed the student's ability to link the situations and formal expressions. The student was presented with the first story situation from one of the sets in Table 1, along with a parentheses-plus or parentheses-minus set of three expressions. The story matched the expression with the parentheses [*a-(b+c)* or *a-(b-c)*]. For example, the first Toy Store situation was presented with the expressions 15-(6+5), 15-6-5, and 15-6+5. The student was asked to look at all the expressions and to decide which one "fits the story best." The interviewer then asked why that expression would fit best, probing as needed to clarify the student's response. For each of the expressions not chosen, the student was asked to change the story to make it fit the expression.

The procedure of choosing the best fit and modifying the situation for the other expressions was repeated for the remaining three settings. Because pilot work had revealed considerable variability in the amount of time required to complete the interview, we always presented the parentheses-plus or parentheses-minus expressions in the money setting (Toy Store or Record Store)

first. We thus insured a full set of responses on these items; many students, especially the younger ones, did not have time to complete the discrete objects situation sets.

Transcribing and Coding

Tape recordings of the interviews were transcribed into verbatim protocols for analysis. Correctness of various responses (e.g., choice of equivalent stories in Phase 1 and equivalence judgments in Phase 2) was coded by at least two coders, who conferred to reach a consensus in cases of disagreement. More qualitative coding schemes were devised to examine other aspects of the data.

Results

Phase 1: Informal Understanding of Principles

The goal of Phase 1 was to assess students' informal understanding of the principles underlying the sign-change rules by having them judge the equivalence of story situations. Table 2 presents the percentages of students at each grade level who correctly chose the equivalent pair of situations for each set in response to the question, "Which of the stories are about the same?" The correctness data were subjected to a repeated-measures analysis of variance, with grade (5, 7, or 9) as a between-subjects variable and setting (discrete objects or money) and expression type (parentheses-plus or parentheses-minus) as within-subjects variables. As would be expected, performance improved with age ($p < .05$), with 77% of the fifth-graders, 88% of the seventh-graders, and 91% of the ninth-graders choosing correct pairs across the four situation sets. The parentheses-minus situation sets (Cookies and Record Store stories) were clearly more difficult for students than were the parentheses-plus sets (Cupcakes and Toy Store stories) ($p < .01$). The effect for story setting was small and nonsignificant.

Insert Table 2 about here

The results just described concerned only the correctness of students' choices of equivalent situations. We also examined students' justifications for their choices, rating for adequacy each student's response to the question, "Why would those be the same?" For a justification to be considered adequate, a student had to point out the features of the situations that made them equivalent because of the composition principles and must not have misrepresented any of the relevant quantities or actions on quantities in the situations. For example, an adequate justification of the equivalence of the Toys situations 1 and 2 (see Table 1) must have included a statement that John bought the same things (a robot car for $6 and a figure for $5) in the two situations. This statement might be elaborated to say that it makes no difference that the toys were bought simultaneously or in sequence. Such a justification indicated that the student was aware of the principle of composition in this setting. Table 3 presents protocol excerpts that represent adequate justifications of equivalence for each of the situation sets.

Insert Table 3 about here

Table 4 presents the percentages of students at each grade level who gave adequate justifications for the equivalence of the story pairs in each set. The percentages of adequate justifications repeat the patterns found for the correctness data. Overall performance improved with age ($p < .01$) and the parentheses-minus situations were more difficult than the parentheses-plus situations ($p < .01$).

Insert Table 4 about here

To summarize performance on Phase 1, students were generally quite successful in judging the equivalence of the story situations and justifying the equivalence in terms of the composition principle (albeit in informal language). They were considerably more successful in judging and justifying the situations represented by the parentheses-plus expressions than the parentheses-minus situations. The parentheses-plus situations can be thought of in terms of successive removal of two sets from a start set or of first combining the two sets to be removed (clearly composition inside parentheses). In the parentheses-minus situations, on the other hand, the set that is removed from the start set must by reduced by a certain amount, so that neither of the sets is removed from the start set in its entirety.

Phase 2: Sign Change Rules Applied to Formal Expressions

In Phase 2 students judged the equality of expression pairs to which the sign change transformations had been correctly and incorrectly applied. Table 5 presents the percentages of students at each grade level who made correct judgments of the equivalence or nonequivalence of each expression pair. The most striking thing about the percentages in Table 5 is the overall low performance on the equal expression pairs. Only 23% of the students correctly judged the expressions 16-(8+3) and 16-8-3 to be equal; 31% of students made correct equivalence judgments of the expressions 11-(5-2) and 11-5+2. These low percentages include the few students who made their correct judgments by calculating the values of the individual expressions in spite of being asked not to calculate. Even the ninth graders, who were taking algebra, did poorly in judging the equivalence of the equal expression pairs.

Insert Table 5 about here

Thus, students' knowledge of the sign-change rule applied to arithmetic expressions was weak. Even after instruction in algebra,

students were not successful in their equivalence judgments of expression pairs.

Phase 3: Linking of Expressions and Situations

Phase 3 focused on students' ability to link the story situations from Phase 1 to formal expressions like those in Phase 2. All the stories presented to the student in Phase 3 were best described by expressions of the form *a-(b+c)* or *a-(b-c)*. The student's first task with each situation was to choose the expression providing the best fit to the story. Because of time constraints many students were not given the opportunity to chose expressions for the Discrete Objects situations (Cupcakes and Cookies); thus only results for the stories in the Money setting (Toy Store and Record Store) are presented. Table 6 displays the presents the percentage of students at each grade level choosing the various expressions as providing the best fit for the Toy Store and Record Store situations. For the Toy Store (parentheses-plus) story, a choice of either or both of the equivalent expressions 15-(6+5) or 15-6-5 was considered correct. Choosing the expressions 14-(8-3) or 14-8+3 was considered correct for the Record Store (parentheses-minus) story. As would be expected, the percentage of students choosing correct expressions increases across grades.

Insert Table 6 about here

For the Toy Store (parentheses-plus) situation, the fifth-graders preferred the expression 15-6-5, the correct expression without parentheses. Apparently these students perceived the purchase of two items in the toy store as two successive transformations to a start set, in other words taking away $6 from the original $15 for purchasing the robot car and removing an additional $5 for the purchase of the little figure. The older students were more likely to pick the expression 15-(6+5), which involves the transformation of the start set by a composed quantity. These students apparently perceived the situation as the reducing of the original $15 by the

combined purchase price of the two items ($6 + $5). For the Record Store (parentheses-minus) situation, students at all grade levels preferred the expression 14-(8-3), the correct expression form containing parentheses. These students apparently perceived the situation as reducing the original $14 by the final price of the record, which was $8 reduced by $3 (8-3). Few of the fifth- and seventh-graders chose the expression 14-8+3, which, though equivalent, did not map as readily onto the situation. By ninth grade, however, more students did choose the correct expression without parentheses, showing some awareness of the transformation of a *reduction* in the price by $3 to an *increase* in the amount of money left with the purchaser.

Students were also asked to justify their choices, describing how the expressions they chose fit the stories. A student's adequate justification of the fit between an expression and a story provides a better indication of the student's understanding of the mapping than a correct choice alone. Table 7 presents the percentage of students at each grade level who were able to give adequate explanations of the match of expressions to the various stories after choosing an expression that did, indeed, fit the situation (the correct expression with or without parentheses). As in Phase 1 students were more successful with the parentheses-plus situations than with the parentheses-minus situations.

Insert Table 7 about here

For the expressions not chosen as ones fitting the situation, the student was asked to modify the situation to match the expression. Thus the student had to link each of the three expressions in each set to a situation: for each expression, the student gave either an explanation of how the expression fit the situation or a modification of the situation to fit the expression. Table 8 presents the percentage of students at each grade level who were were able to adequately describe the match of each expression to the story or to modify the

story appropriately. The percentages here are lower than in Table 7, indicating that some students were able to choose an appropriate expression and give an adequate justification of its fit to the story situation, but not to modify the story to fit the other expressions.

Insert Table 8 about here

The students' attempts to justify the fit of expressions to situations and to modify the situations resulted in an interesting array of errors. Many of the errors reflected difficulties in conceiving of successive transformations to a quantity. For example, for the expression 15-6+5, one ninth-grader modified the Toy Store story as follows: "John went to the toy store with $15 and he bought a robot car for 6, and he was left with 5. But he had $5 left over." This student was unable to incorporate the second transformation (+5) appropriately into the story, resulting in its interpretation as a state (amount of money left over) instead of a transformation (amount of money removed from the start set).

The largest categories of erroneous modifications appeared to be a result of students treating expressions in a strictly linear and localized fashion, rather than conceiving of the entire expression as representing a quantity. For example, in a typical modification of the Record Store situation to fit the expression 14-8+3, one ninth-grade student said "Sally went to the record [store] with $14 and bought a record; the record was usually $8, but is was marked higher; it was raised, the price was raised $3." This student correctly interpreted the +3 as an increase ("it was raised"), but applied the increase to the wrong quantity--the price of the record instead of the amount of money Sally has. The student did not seem able to construct an adequate representation of the two successive transformations in the expression. Errors like this were made by numerous students on all of the stories and expression types. They had considerable difficulty constructing appropriate situations for

expressions, again reflecting difficulties in linking the formal symbols with the reference domains represented in the situations.

Even after matching expressions to and modifying the situations in Phase 3, few students were able to justify the equivalence of the expressions in terms of the stories. The potential power of thinking of the expressions in terms of situational referents is, however, illustrated by the students who were successful in explaining the equivalence of the expressions in terms of the situations. For example, one seventh-grader in Phase 2 had declared the expressions 16-(8+3) and 16-8-3 to be not equal because in the first "you're saying 16 minus 8 plus 3" and in the second "you're saying 16 minus 8 minus 3." Note that the student seemed to be ignoring the parentheses, focusing only on the fact that there are different operations in the two expressions. After some false starts matching expressions of the same type to the Cupcake story in Phase 3 (Cupcake story 1 above), this student correctly chose and justified the expression 18-(7+2) as best fitting the story, and modified the story appropriately for the expressions 18-7-2 ("he sold 7 yellow cupcakes and the next day he went to another bake sale and sold 2") and 18-7+2 ("He sold 7 cupcakes at the bake sale, and then when he got home, his mother baked him 2 more."). When subsequently shown just the three expressions [18-(7+2), 18-7-2, and 18-7+2] and asked if any of them "would come out to be equal," the student correctly said that 18-(7+2) and 18-7-2 would be equal "because in each story David sold the same amount of cupcakes out of 18, so naturally it's going to come out the same answer." By mapping the symbols in the expressions to quantities in the situations, it has become obvious to this student that the two expressions are equivalent. The student was thus able to link the formal symbolic expressions to his more intuitive knowledge about how quantities behave in situations.

Conclusion

The ultimate goal of this line of research is to develop ways to improve students' understanding of the symbolic manipulations they learn in algebra. We believe that increasing students' understanding

of referential meaning of formal symbol systems may facilitate the learning of formal rules and the application of algebra to problem solving and learning more advanced mathematics. This study provides important psychological description needed as a base for the development of instructional interventions.

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Table 1
Story Situations

Parentheses-Plus Situations

Discrete Object Setting (Cupcakes)

1. David took 18 cupcakes to the bake sale. He sold 7 chocolate ones and 2 yellow ones.
   18-(7+2)

2. David took 18 cupcakes to the bake sale. He sold 7 chocolate ones.

   Then David's mother brought him 2 more yellow cupcakes.
   18-7+2

3. David took 18 cupcakes to the bake sale. At lunch time he sold 7 chocolate ones.

   After school he sold 2 yellow ones.
   18-7-2

Money Setting (Toy Store)

1. John went to the toy store with $15. He bought a robot car for $6 and a little figure for $5.
   15-(6+5)

2. John went to the toy store with $15. He bought a robot car for $6.

   Then John saw a little figure he wanted. He bought it for $5.
   15-6-5

3. John went to the toy store with $15. He bought a robot car for $6.

   Then John's mother gave him $5 allowance.
   15-6+5

Parentheses-Minus Situations

Discrete Objects Setting (Cookies)

1. Cathy had 31 cookies. She was going to take 16 heart cookies for her class party. But 4 of the hearts had cinnamon candies and she did not like them. So Cathy left those 4 at home.
   31-(16-4)

2. Cathy baked 31 cookies. She took 16 heart cookies to school. But 4 cookies weren't eaten because the cookies had cinnamon candies on them and no one liked them. So Cathy brought those 4 back home.
   31-16+4

3. Cathy baked 31 cookies. She put 16 heart cookies in a box for school. Then she put 4 more cookies with cinnamon candies in another box for her aunt.
   31-16-4

Money Setting (Record Store)

1. Sally went to the record store with $14 and bought a record. The record was usually $8 but was marked $3 off.
   14-(8-3)

2. Sally went to the record store with $14. She bought a record for $8.

   On her way out Sally saw another record she wanted to buy. She bought it for $3.
   14-8-3

3. Sally went to the record store with $14. She bought a record for $8.

   After Sally paid for the record, she remembered she had a $3 gift certificate. So, the clerk gave her $3 in cash for it.
   14-8+3

Table 2
Phase 1: Percentages of Students Choosing Correct Story Pairs as Equivalent

Situation Sets	Grade 5	7	9	Means
Parentheses-Plus				
Money Setting (Toy Store)	96	100	100	99
Discrete Objects Setting (Cupcakes)	79	100	100	93
Parentheses-Minus				
Money Setting (Record Store	55	61	82	66
Discreate Objects Setting (Cookies)	79	86	82	82
Means	77	87	91	

Table 3

Examples of Adequate Jusfitications for Equivalent Situations in Phase 1

Toys (Stories 1 and 2 are equivalent)

Seventh-Grade Student (710)

. . . Can you tell me how you decided? **Well, he ends up with $4 left in number 1 and number 2.** Okay. If you weren't figuring out the answer, is there anything in the story that would tell you? **Well, he bought the robot car and a little figure in both of them.** Okay. These stories [1 and 2] are a little bit different. Can you tell me about that? **Um, well in the first one, right away he bought the robot car and a little figure. And in the second one, he bought a robot car and then after that he saw a little figure he wanted.** Okay. Does it make any difference that they're bought at different times? **Not really.**

Records (Stories 1 and 3 are equivalent)

Seventh-Grade Student (712)

1 and 3? How did you decide that? **Well, the record store, they're boght giving her $3 off her record.** Hmm. **This one [Story 1] is marked off $3. In this one [Story 3] they gave her a certificate for it.** I thought she gave them the certificate. **Oh, yeah, and then . . . they gave her the cash.** So that meant? **That both 1 and 3, . . . she was getting a refund for it.**

Cupcakes (Stories 1 and 3 are equivalent)

Fifth-Grade Student (502)

Which of those are about the same? **Um, number 1 and 3 are the same because he, on each of them, he sold 9 cookies. And he sold the same amount on them. Becuase. . . on card 1 he sold 7 chocolate ones and 2 yellow ones in the same time period. And on number 3, he sold 7 chocolate ones before lunch, I mean at lunch time. And then after school he sold 2 yellow ones.**

Cookies (Stories 1 and 2 are equivalent)

Ninth-Grade Student (902)

1 and 2 are the same because it would be the same, she took them to school and then she brought them back home. And that would be, like, the same as leaving them at home, instead of taking them to school.

Table 4
Phase 1: Percentages of Students Giving Adequate Justifications of Equivalent Story Pairs

Situation Sets	Grade 5	7	9	Means
Parentheses - Plus				
Money Setting (Toy Store)	86	93	89	89
Discrete Objects Setting (Cupcakes)	74	96	85	85
Parentheses-Minus				
Money Setting (Record Store)	43	64	79	62
Discrete Objects Setting (Cookies)	26	48	68	47
Means	57	75	80	

Table 5

Phase 2: Percentage (frequency) of Students Correctly Judging Equivalence or Non-equivalence of Expressions

Expressions	Grade 5	7	9
16 - (8 + 3) (parentheses-plus)			
Equal to 16 - 8 - 3	18(5)**	28(8)*	21(6)**
Not Equal to 16 - 8 + 3	57(16)	32(9)	79(22)**
Not Equal to 16 + 8 - 3	68(19)	93(26)**	79(22)**
11 - (5 - 2) (parenthesis-minus)			
Equal to 11 - (5 - 2)	28(8)*	14(4)**	46(13)
Not Equal to 11 - 5 - 2	57(16)	50(14)	79(22)**
Not Equal to 11 + 5 - 2	93(26)**	96(27)**	96(27)**
	n = 28	n = 28	n = 28

*$p < .05$ **$p < .01$

Table 6
Phase 3: Percentage (Frequency) of Students Choosing Each Expression as Providing Best Fit to Situation

Expressions	Grade 5	Grade 7	Grade 9
Toy Store (parentheses-plus)			
15-(6+5) correct w/ parens.	42(11)	50(13)	61(17)
15-6-5 correct w/o parens.	46(12)[a]	46(12)	39(11)[a]
total correct	88(23)		
15-6+5 incorrect	12(3)	4(1)	--
Record Store (Parenthes-minus)			
14-(8-3) correct w/ parens.	54(13)[a]	64(16)	46(13)[a]
14-8+3 correct w/o parens.	17(4)	16(4)	39(11)[b]
total correct	71(17)	80(20)	85(24)
14-8-3 incorrect	25(6)	20(5)	14(4)[a]
"none fits"	4(1)	--	--

[a] Includes one student who said two expressions fit equally well.

[b] Includes two students who said two expressions fit equally well.

Table 7
Phase 3: Percentages of Students Giving Adequate Justifications of Best Fit Expressions

Situations	Grades 5	7	9	Means
Toy Store (parentheses-plus)	77	85	93	85
Record Store (parentheses-minus)	54	76	86	72
Means	66	81	90	

Table 8
Phase 3: Percentages (frequencies) of Students Giving Adequate Modification of Situation or Justification of Fit for Each Expression

Expressions	Grade 5	7	9
Toy Store			
15-(6+5)	58(14)	84(21)	79(22)
15-6-5	63(15)	85(22)	82(23)
15-6+5	52(12)	73(19)	75(21)
Record Store (Par.)			
14-(8-3)	60(12)	79(19)	64(18)
14-8+3	35(8)	57(16)	81(22)
14-8-3	55(12)	65(15)	56(15)

Note. Ns vary because of missing data; some students were not asked to modify some expressions.